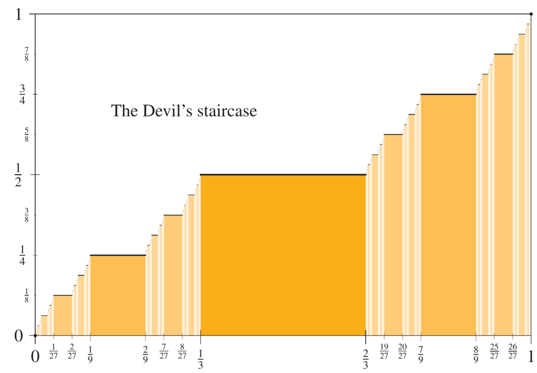
Advanced Real Analysis 1

math 564 Anush Tserunyan 2023 fall

TueThu 16:05–17:25 Location: BURN 1205



Cantor's function

Real analysis before 20th century as taught at the undergraduate level in terms of Riemann integration and continuous/differentiable functions, is not adequate for the modern needs of functional analysis, differential equations, probability theory, descriptive set theory, and so on. Its main shortcoming is the restrictedness of the class of functions to which it applies: for example, a pointwise limit of a sequence of continuous (or even smooth) functions on $[0,1]$ may already *not* be Riemann integrable. Measure theory fixes this issue, enlarging the class of integrable functions so that it is closed under countable operations such as pointwise limits. In this course we develop abstract measure theory and measure differentiation, and study the relevant spaces of functions and their duals. The course will also give hints of flavours of ergodic theory and descriptive set theory.

Prerequisites: MATH 454 and MATH 455 are the official prerequisits. Unofficially, MATH 254 and MATH 255 should suffice, as long as the students are comfortable with convergence of sequences and functions, continuity, basic pointset topology notions (open/closed sets, boundary), and compactness. Having seen abstract metric spaces and the Heine–Borel theorem would help a lot.

Assessment: 30% homework + 30% midterm + 40% final. There will be 6 homework assignments (one every two weeks), to be submitted on Crowdmark. The midterm will be held in the week of Oct 16. The date of the final will be announced later.

References

- T. Tao's blog: courses 245A and 245B
- R. Bass, *Real Analysis for Graduate Students* [free download]
- G. Folland, *Real Analysis* [link] (not required)

Topics

- *Measures, their construction and properties*

  (1) Polish spaces, σ-algebras and Borel sets, measurable spaces

  (2) Measures and premeasures

(3) Constructions of Bernoulli measures on $2^{\mathbb{N}}$ and the Lebesgue measure on $\mathbb{R}^d$

(4) Carathéodory's extension theorem: outer measures and two different proofs (by C. Carathéodory and T. Tao)

(5) Measurable and non-measurable sets

(6) Pocket tools: increasing unions/decreasing intersections, Borel–Cantelli lemmas, measure exhaustion and application: Sierpiński's theorem for atomless measures

(7) Borel measures: their regularity (for metric spaces) and tightness (for Polish spaces), the 99% lemma for Bernoulli(p) and Lebesgue measures

(8) Applications: ergodic group actions and non-measurability of transversals

(9) Locally finite Borel measures on $\mathbb{R}$ and increasing right-continuous functions

- *Measurable functions and integration*

(10) Measurable functions, Luzin's theorem, push-forward measures, and random objects such as random graphs

(11) Borel and measure isomorphism theorems (sketches of proofs)

(12) Simple functions and their integration, approximation of measurable functions by simple ones

(13) Integration of non-negative functions, monotone convergence theorem, and Fatou's lemma

(14) Integration of real/complex valued functions and L^1, dominated convergence theorem, and the density of simple functions in L^1

(15) Properties of integrable functions: σ-finiteness of the support, 99% boundedness, absolute continuity of $B \mapsto \int_B f \, d\mu$

(16) Convergence in measure and relations between different modes of convergence

(17) Product measures and the Fubini–Tonelli theorem

(18) (Optional) Introduction to ergodic theory and the proof of the pointwise ergodic theorem

- *Measure differentiation, density, and a.e. differentiable functions*

(19) Orthogonality and absolute continuity of measures, Jordan decomposition

(20) Signed measures and Hahn decomposition (proof via measure exhaustion)

(21) The Lebesgue–Radon–Nikodym theorem and Radon–Nikodym derivatives

(22) The Lebesgue differentiation theorem for $\mathbb{R}^d$: proof via the Hardy–Littlewood maximal function and the Vitali covering lemma

(23) The Lebesgue density theorem for $\mathbb{R}^d$, Lebesgue differentiation theorem for all locally finite Borel measures on $\mathbb{R}^d$

(24) Characterization of the distribution functions on $\mathbb{R}$ whose associated measures are absolutely continuous/orthogonal with respect to Lebesgue measure

(25) Absolutely continuous functions and the fundamental theorem of calculus for increasing functions

(26) Finite Borel signed measures on $\mathbb{R}$ and right-continuous functions of bounded variation, the fundamental theorem of calculus for functions of bounded variation

- *L^p spaces*

(27) Norms and Banach spaces, L^1 is a Banach space

(28) L^p norm and Minkowski's inequality

(29) Hölder's inequality and relations between L^p spaces

(30) L^2 as *the* Hilbert space (only statements, no proofs)

(31) Bounded linear transformations and functionals, equivalence of continuity and Lipschitzness, operator norm

(32) The dual of L^p for $1 \le p < \infty$ (sketch of proof)